# On the Skeleton of Stonian p-Ortholattices

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**Abstract.** Boolean Contact Algebras (BCA) establish the algebraic counterpart of the mereotopolopy induced by the Region Connection Calculus (RCC). Similarly, Stonian p-ortholattices serve as a lattice theoretic version of the ontology $RT^{-}$ of Asher and Vieu. In this paper we study the relationship between BCAs and Stonian p-ortholattices. We show that the skeleton of every Stonian p-ortholattice is a BCA, and, conversely, that every BCA is isomorphic to the skeleton of a Stonian p-ortholattice. Furthermore, we prove the equivalence between algebraic conditions on Stonian p-ortholattices and the axioms C5, C6, and C7 for BCAs.

## 1 Introduction

Region-based theories of space play a crucial role in qualitative spatial reasoning (QSR) within Artificial Intelligence (cf. [4]). Mereotopology – consisting of some topological notion of *contact* and a mereological notion of *parthood* – is the common core to most region-based theories of space. Instead of points as in classical point-set topology, mereotopology uses regions as primitives and focuses on the qualitative relations between different regions, such as contact, overlap, external contact, and parthood. In allowing to define part-whole relations such as self-connectedness of regions, the combination of topology with mereology is more expressive than either theory by itself.

As long as AI has been interested in mereotopology, different first-order mereotopological theories have been proposed. Most prominent amongst them is the Region-Connection Calculus (RCC) [3], which originated from Clarke's theory [2]. Another theory of the same origin, the $RT_0$ by Asher and Vieu [1], has received

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less attention although the theory is fairly similar to the RCC. The two theories differ mainly in their intended topological interpretations: RCC models include only regular closed sets while RT models allow any kind of regular sets (closed, open, clopen, or neither). A very fruitful way of understanding these theories of qualitative space is by looking at their algebraic counterparts. For the RCC, it was shown that the models can be defined in terms of Boolean Contact Algebras (BCA) for which topological representations have been given for various subsets of the original RCC axioms [6, 7, 11, 12]. Recently, an algebraic representation of the theory $\mathrm{RT}^-$ as Stonian p-ortholattices [10] has been proved which allows to compare the models of RCC and $\mathrm{RT}^-$ in a purely algebraic way.

In this work, we exhibit the relationship between BCAs and Stonian p-ortholattices by using the skeleton of Stonian p-ortholattices as bridging structure. We show that the skeleton $S(L)$ of an arbitrary Stonian p-ortholattice $L$ is a BCA when defining the contact relation of the BCA in terms of the lattice $L$. In addition we prove the equivalence between algebraic conditions on Stonian p-ortholattices and the axioms C5, C6, and C7 for BCAs. On the reverse, we prove that every BCA can be embedded in a Stonian p-ortholattice. This theoretical work provides semantic mappings between the two theories; it specifies which class of models of the $\mathrm{RT}^-$ can be mapped to which class of BCAs and vice versa.

The paper is structured as following. Section 2 introduces Stonian p-ortholattices and their algebraic properties. We define standard topological models and the notion of a skeleton for Stonian p-ortholattices. Afterwards, we briefly review BCAs and their topological representation. The following two sections contain the main results of this paper. In Section 4 we establish that the skeleton of a Stonian p-ortholattice is a BCA when choosing the contact relation accordingly, and in Section 5 we construct a Stonian p-ortholattice from any BCA by using the Boolean algebra of a BCA as the skeleton of the Stonian p-ortholattice. However, examples demonstrate that there is no unique embedding of BCAs into Stonian p-ortholattices.

These constructive embedding theorems verify in an algebraic way that the models of the theory $\mathrm{RT}^-$ are indeed more general than BCAs. Most significantly for QSR, the results imply that every model of $\mathrm{RT}^-$ that is connected, $*$-normal, and has a dense skeleton is in fact a model of the RCC. However, arbitrary Stonian p-ortholattices $L$ of $\mathrm{RT}^-$ models do not adhere to the extensionality, interpolation, and connection axioms. Their skeletons $S(L)$ are arbitrary BCAs as axiomatized by C0-C4.

## 2 Stonian p-Ortholattices

In [1] Asher and Vieu introduced the mereotopology $\mathrm{RT}_0$. This theory was intended to cover exactly those regions that have full interior and smooth boundaries. Even though this theory does not include all possible sets of a topological space, the notion of interior and closure are available. In [10] it was shown that the models of Asher and Vieu's theory $\mathrm{RT}^-$ are equivalent to Stonian (or Stonean) p-ortholattices. This observation now allows an algebraic treatment of that theory. First, recall pseudocomplemented and orthocomplemented lattices.

**Definition 1.** *A pseudocomplemented lattice (or p-algebra) is an algebraic structure $\langle L, +, \cdot, ^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that*

P0. $\langle L, +, \cdot, 0, 1\rangle$ *is a bounded lattice,*
P1. $a^*$ *is the pseudocomplement of $a$, i.e.* $a \cdot x = 0 \iff x \leq a^*$.

**Definition 2.** *An ortholattice (or orthocomplemented lattice) is an algebraic structure $\langle L, +, \cdot, ^\perp, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that*

O0. $\langle L, +, \cdot, 0, 1\rangle$ *is a bounded lattice,*
O1. $a^\perp$ *is an orthocomplement of $a$, i.e. for all $a, b \in L$ we have*
  *(a)* $a^{\perp\perp} = a$,
  *(b)* $a \cdot a^\perp = 0$,
  *(c)* $a \leq b$ *implies* $b^\perp \leq a^\perp$.

A lattice that is both pseudocomplemented and orthocomplemented is called a p-ortholattice. A Stonian p-ortholattice additionally satisfies the Stone identity (PO.2). Notice that p-ortholattices are not necessarily distributive. In fact any distributive Stonian p-ortholattice is a Boolean algebra (cf. [10]).

**Definition 3.** *A Stonian p-ortholattice is an algebraic structure $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that*

PO0. $\langle L, +, \cdot, ^*, 0, 1\rangle$ *is a pseudocomplemented lattice,*
PO1. $\langle L, +, \cdot, ^\perp, 0, 1\rangle$ *is an ortholattice,*
PO2. $(a \cdot b)^* = a^* + b^*$ *holds for all $a, b \in L$.*

P-ortholattices are always quasicomplemented (also known as 'dually pseudocomplemented') and thus double p-algebras. In a Stonian p-ortholattice one may define the quasicomplement $a^+$ of $a$, i.e. the smallest element $b$ such that $a + b = 1$, as $a^+ = a^{\perp * \perp}$.

The following basic properties of Stonian p-ortholattices were shown in [10].

**Lemma 1.** *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice. Then:*

1. $0^+ = 0^\perp = 0^* = 1$ *and* $1^+ = 1^\perp = 1^* = 0$,
2. $a \cdot a^+ = a \cdot a^\perp = a \cdot a^* = 0$ *and* $a + a^+ = a + a^\perp = a + a^* = 1$,
3. $a^+ \leq a^\perp \leq a^*$ *and* $a^{++} \leq a \leq a^{**}$
4. $a^{+++} = a^+$ *and* $a^{***} = a^*$,
5. $a \leq b$ *implies* $b^* \leq a^*$, $b^+ \leq a^+$, *and* $b^\perp \leq a^\perp$,
6. $(a+b)^* = a^* \cdot b^*$, $(a \cdot b)^+ = a^+ + b^+$, $(a+b)^\perp = a^\perp \cdot b^\perp$ *and* $(a \cdot b)^\perp = a^\perp + b^\perp$,
7. $a^{*\perp} = a^{\perp +} = a^{*+} = a^{++}$ *and* $a^{+\perp} = a^{\perp *} = a^{+*} = a^{**}$.

Throughout this paper we will use the properties above without mentioning.

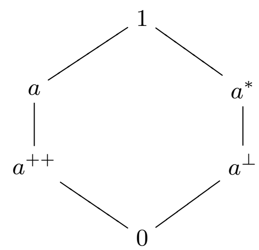


**Fig. 1.** The non-modular, non-distributive Stonian p-ortholattice $C_6$

### 2.1 Topological models

Topological models of the theory of Stonian p-ortholattices are given by those sets that have full interior and smooth boundaries, i.e. are based on $\mathrm{RT}(X) = \{a \subseteq X \mid \mathrm{int}(a) = \mathrm{int}(\mathrm{cl}(a)) \wedge \mathrm{cl}(a) = \mathrm{cl}(\mathrm{int}(a))\}$ where int and cl are the interior and closure operation of the topological space $\langle X, \tau \rangle$. On those elements we define the following operations. The notations $x \cap^* y$ and $x \cup^* y$ are maintained from [1].

$$\begin{aligned}
x \cap^* y &= x \cap y \cap \mathrm{cl}(\mathrm{int}(x \cap y)), \\
x \cup^* y &= x \cup y \cup \mathrm{int}(\mathrm{cl}(x \cup y)), \\
x^* &= \mathrm{cl}(X \setminus x), \\
x^\perp &= X \setminus x.
\end{aligned}$$

The next lemma provides some basic properties of those operations.

**Lemma 2.** *Let $\langle X, \tau \rangle$ be a topological space. Then we have:*

1. $\mathrm{cl}(x \cup^* y) = \mathrm{cl}(x) \cup \mathrm{cl}(y)$,
2. $\mathrm{int}(x \cap^* y) = \mathrm{int}(x) \cap \mathrm{int}(y)$,
3. $X \setminus (x \cap^* y) = (X \setminus x) \cup^* (X \setminus y)$,
4. $X \setminus (x \cup^* y) = (X \setminus x) \cap^* (X \setminus y)$.

*Proof.* 1. Consider the following computation

$$\begin{aligned}
\mathrm{cl}(x \cup^* y) &= \mathrm{cl}(x \cup y \cup \mathrm{int}(\mathrm{cl}(x \cup y))) \\
&= \mathrm{cl}(x \cup y) \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(x \cup y))) \\
&= \mathrm{cl}(x \cup y) && \mathrm{cl}(\mathrm{int}(\mathrm{cl}(z))) \subseteq \mathrm{cl}(z) \\
&= \mathrm{cl}(x) \cup \mathrm{cl}(y).
\end{aligned}$$

2. is shown analogously.

3. This property is shown by

$$\begin{aligned}
X \setminus (x \cap^* y) &= X \setminus (x \cap y \cap \mathrm{cl}(\mathrm{int}(x \cap y))) \\
&= (X \setminus x) \cup (X \setminus y) \cup (X \setminus \mathrm{cl}(\mathrm{int}(x \cap y))) \\
&= (X \setminus x) \cup (X \setminus y) \cup \mathrm{int}(\mathrm{cl}(X \setminus (x \cap y))) \\
&= (X \setminus x) \cup (X \setminus y) \cup \mathrm{int}(\mathrm{cl}((X \setminus x) \cup (X \setminus y))) \\
&= (X \setminus x) \cup^* (X \setminus y).
\end{aligned}$$

4. is shown analogously. □

The next theorem verifies that the class of all structures $\mathrm{RT}(X)$ can be seen as the class of standard topological models of this kind of mereotopology.

**Theorem 1.** *Let $\langle X, \tau \rangle$ be a topological space. Then $\langle \mathrm{RT}(X), \cup^*, \cap^*, {}^*, {}^\perp, \emptyset, X \rangle$ is a Stonian p-ortholattice.*

*Proof.* First, we have to show that $\mathrm{RT}(X)$ is closed under all operations. Consider the following computations

$$\begin{aligned}
\mathrm{cl}(x \cup^* y) &= \mathrm{cl}(x) \cup \mathrm{cl}(y) && \text{Lemma 2(1)} \\
&= \mathrm{cl}(\mathrm{int}(x)) \cup \mathrm{cl}(\mathrm{int}(y)) && x, y \in \mathrm{RT}(X) \\
&= \mathrm{cl}(\mathrm{int}(x) \cup \mathrm{int}(y)) \\
&\subseteq \mathrm{cl}(\mathrm{int}(x \cup y)) \\
&\subseteq \mathrm{cl}(\mathrm{int}(x \cup^* y)),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{int}(\mathrm{cl}(x \cup^* y)) \\
&= \mathrm{int}(\mathrm{cl}(x) \cup \mathrm{cl}(y)) && \text{Lemma 2(1)} \\
&= \mathrm{int}(\mathrm{int}(\mathrm{cl}(x) \cup \mathrm{cl}(y))) \\
&= \mathrm{int}(\mathrm{int}(\mathrm{cl}(x)) \cup \mathrm{int}(\mathrm{cl}(y)) \cup \mathrm{int}(\mathrm{cl}(x) \cup \mathrm{cl}(y))) && \mathrm{int}(z_1) \cup \mathrm{int}(z_2) \\
& && \quad \subseteq \mathrm{int}(z_1 \cup z_2) \\
&= \mathrm{int}(\mathrm{int}(x) \cup \mathrm{int}(y) \cup \mathrm{int}(\mathrm{cl}(x) \cup \mathrm{cl}(y))) && x, y \in \mathrm{RT}(X) \\
&= \mathrm{int}(\mathrm{int}(x) \cup \mathrm{int}(y) \cup \mathrm{int}(\mathrm{cl}(x \cup y))) \\
&\subseteq \mathrm{int}(x \cup y \cup \mathrm{int}(\mathrm{cl}(x \cup y))) \\
&= \mathrm{int}(x \cup^* y).
\end{aligned}$$

In both cases the converse inclusion is trivial. The properties $\mathrm{int}(x \cup^* y) = \mathrm{int}(\mathrm{cl}(x \cup^* y))$ and $\mathrm{int}(x \cap^* y) = \mathrm{int}(\mathrm{cl}(x \cap^* y))$ are shown analogously.

$$\begin{aligned}
\mathrm{cl}(x^\perp) &= \mathrm{cl}(X \setminus x) \\
&= X \setminus \mathrm{int}(x) \\
&= X \setminus \mathrm{int}(\mathrm{cl}(x)) && x \in \mathrm{RT}(X) \\
&= \mathrm{cl}(\mathrm{int}(X \setminus x)) \\
&= \mathrm{cl}(\mathrm{int}(x^\perp)),
\end{aligned}$$

$$
\begin{aligned}
\mathrm{int}(x^\perp) &= \mathrm{int}(X \setminus x) \\
&= X \setminus \mathrm{cl}(x) \\
&= X \setminus \mathrm{cl}(\mathrm{int}(x)) && x \in \mathrm{RT}(X) \\
&= \mathrm{int}(\mathrm{cl}(X \setminus x)) \\
&= \mathrm{int}(\mathrm{cl}(x^\perp)), \\
\mathrm{cl}(x^*) &= \mathrm{cl}(\mathrm{cl}(X \setminus x)) \\
&= \mathrm{cl}(X \setminus x) \\
&= \mathrm{cl}(x^\perp) \\
&= \mathrm{cl}(\mathrm{int}(x^\perp)) && \text{see above} \\
&= \mathrm{cl}(\mathrm{int}(X \setminus x)) \\
&\subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(X \setminus x))) \\
&= \mathrm{cl}(\mathrm{int}(x^*)), \\
\mathrm{int}(x^*) &= \mathrm{int}(\mathrm{cl}(X \setminus x)) \\
&= \mathrm{int}(\mathrm{cl}(\mathrm{cl}(X \setminus x))) \\
&= \mathrm{int}(\mathrm{cl}(x^*)).
\end{aligned}
$$

Now, assume $x, y, z \in \mathrm{RT}(X)$ with $z \subseteq x$ and $z \subseteq y$. Then $z \subseteq x \cap y$, and we have $z = z \cap \mathrm{cl}(z) = z \cap \mathrm{cl}(\mathrm{int}(z)) \subseteq x \cap y \cap \mathrm{cl}(\mathrm{int}(x \cap y)) = x \cap^* y$. This verifies that $x \cap^* y$ is the greatest lower bound of $x$ and $y$ in $\mathrm{RT}(X)$. It is shown analogously that $x \cup^* y$ is the least upper bound of $x$ and $y$ in $\mathrm{RT}(X)$.

It is easy to verify that $x^\perp$ is an orthocomplement of $x$. In order to prove that $x^*$ is a pseudocomplement consider the following computation

$$
\begin{aligned}
x \cap^* x^* &= x \cap \mathrm{cl}(X \setminus x) \cap \mathrm{cl}(\mathrm{int}(x \cap \mathrm{cl}(X \setminus x))) \\
&\subseteq \mathrm{cl}(\mathrm{int}(x \cap \mathrm{cl}(X \setminus x))) \\
&= \mathrm{cl}(\mathrm{int}(x) \cap \mathrm{int}(\mathrm{cl}(X \setminus x))) \\
&= \mathrm{cl}(\mathrm{int}(x) \cap \mathrm{int}(X \setminus x)) && X \setminus x \in \mathrm{RT}(X) \\
&= \mathrm{cl}(\mathrm{int}(x \cap (X \setminus x))) \\
&= \mathrm{cl}(\mathrm{int}(\emptyset)) \\
&= \emptyset.
\end{aligned}
$$

In order to verify that $x^*$ is the pseudocomplement of $X$ it remains to show that $x^*$ is the largest element $z$ with $x \cap^* z = \emptyset$. Therefore, assume $z \in \mathrm{RT}(X)$ with $x \cap^* z = \emptyset$. Then we have $\mathrm{int}(x) \cap \mathrm{int}(z) = \mathrm{int}(x \cap^* z) = \mathrm{int}(\emptyset) = \emptyset$ using Lemma 2(2). We conclude $\mathrm{int}(z) \subseteq X \setminus \mathrm{int}(x) = \mathrm{cl}(X \setminus x) = x^*$. This immediately implies $z \subseteq \mathrm{cl}(z) = \mathrm{cl}(\mathrm{int}(z)) \subseteq \mathrm{cl}(x^*) = x^*$.

The following computation verifies the Stone property

$$
\begin{aligned}
(x \cap^* y)^* &= \mathrm{cl}(X \setminus (x \cap^* y)) \\
&= \mathrm{cl}((X \setminus x) \cup^* (X \setminus y)) && \text{Lemma 2(3)} \\
&= \mathrm{cl}(X \setminus x) \cup \mathrm{cl}(X \setminus y) && \text{Lemma 2(1)} \\
&= x^* \cup y^* \\
&= x^* \cup y^* \cup \mathrm{int}(x^* \cup y^*) \\
&= x^* \cup y^* \cup \mathrm{int}(\mathrm{cl}(X \setminus x) \cup \mathrm{cl}(X \setminus y)) \\
&= x^* \cup y^* \cup \mathrm{int}(\mathrm{cl}(\mathrm{cl}(X \setminus x) \cup \mathrm{cl}(X \setminus y))) \\
&= x^* \cup y^* \cup \mathrm{int}(\mathrm{cl}(x^* \cup y^*)) \\
&= x^* \cup^* y^*.
\end{aligned}
$$

This completes the proof. □

### 2.2 Skeleton

Skeletons (also called centers) have been first defined by Glivenko in 1929 for Brouwerian lattices [9], showing that pseudocomplementation is a closure mapping. Frink [8] generalized this result by showing that the skeleton of a pseudocomplemented meet-semilattices is always a Boolean algebra.

**Definition 4.** *Let $\langle L, \cdot, ^*, 0\rangle$ be a pseudocomplemented semilattice. Let $S(L) = \{a^* | a \in L\}$ be the skeleton of $L$, maintaining the order relation of $L$ and with meet $a \wedge b = a \cdot b$ and union $a \vee b = (a^* \cdot b^*)^*$.*

**Theorem 2 (Glivenko-Frink Theorem).** *[8] Let $L$ be a pseudocomplemented semilattice. Then $S(L)$ is a Boolean algebra. The (unique) complement of an element $a \in S(L)$ is its pseudocomplement $a^* \in L$.*

Since Stonian p-ortholattices form a subclass of the class of pseudocomplemented meet-semilattices, the previous theorem immediately implies the following corollary (cf. [10]). Notice that here we have a stronger notion: the skeleton is not just a Boolean algebra, but a Boolean subalgebra.

**Corollary 1.** *If $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ is a Stonian p-ortholattice, then $S(L)$ is a Boolean subalgebra of $L$.*

### 2.3 Additional Properties of Stonian p-Ortholattices

Motivated by the topological interpretation of the operations (cf. [10]), we call an element $a \in S(L)$, i.e. an element with $a^{**} = a$, closed. Dually, we call $a$ open if $a^{++} = a$, and clopen if it is open and closed.

$L$ is called connected iff $0, 1$ are the only clopen elements of $L$.

A topological space is called normal if any two disjoint closed sets can be separated by disjoint open sets. Following this definition we call a Stonian p-ortholattice $L$ $*$-normal if for all $a, b \in L$ with $a^{**} \leq b^{+}$ there is an element $c \in L$ with $a^{**} \leq c^{++}$ and $b^{**} \leq c^{+}$. Notice that in this case $c^{+} = c^{\perp\perp+} = c^{\perp++}$. Then $c \cdot c^{\perp} = 0$ implies $c^{++} \cdot c^{\perp++} = 0$ and hence $c^{++} \cdot c^{+} = 0$, ensuring that the open sets $c^{+}$ and $c^{++}$ are disjoint.

A bounded sublattice $L'$ of $L$ is called (downwards) dense in $L$ if for every $0 \neq a \in L$ there is a $0 \neq b \in L'$ with $b \leq a$.

In Section 4 we are going to show that denseness, $*$-normality and connectedness correspond to well-known additional properties of Boolean contact algebras. But beforehand, we review Boolean contact algebras and their embedding into the Boolean algebra of regular closed sets of a topological space.

## 3 Boolean Contact Algebras

Boolean contact algebras were introduced as the algebraic counterpart of mereotopologies induced by the *Region Connection Calculus* RCC [3]. Therefore, they are intended to cover closed sets with full interior and smooth boundaries, i.e. regular closed sets.

**Definition 5.** *A binary relation $C$ on a Boolean algebra $\langle B, +, \cdot, ^{*}, 0, 1\rangle$ is called a* contact relation *if it satisfies:*

C0. $(\forall a) 0(-C)a$;
C1. $(\forall a)[a \neq 0 \Rightarrow aCa]$;
C2. $(\forall a)(\forall b)[aCb \Rightarrow bCa]$;
C3. $(\forall a)(\forall b)(\forall c)[(aCb \wedge b \leq c) \Rightarrow aCc]$;
C4. $(\forall a)(\forall b)(\forall c)[aC(b + c) \Rightarrow (aCb \vee aCc)]$.

*The pair $\langle B, C\rangle$ is called a* Boolean Contact Algebra *(BCA).*

Additionally, the following properties are of importance:

C5. $(\forall a)(\forall b)[(\forall c)(aCc \Rightarrow bCc) \Leftrightarrow a = b]$. (The extensionality axiom).
C6. $(\forall a)(\forall b)[(\forall c)(aCc \vee bCc^{*}) \Rightarrow aCb]$ (The interpolation axiom).
C7. $(\forall a)[(a \neq 0 \wedge a \neq 1) \Rightarrow aCa^{*}]$ (The connection axiom).

As shown in [12], in the presence of the other axioms we can replace C5 by

C5'. $(\forall a \neq 1)(\exists b \neq 0)[a(-C)b]$.

As already mentioned above, the standard models of Boolean contact algebras are given by the regular closed sets of a topological space together with the following operations:

$$\begin{aligned} x + y &:= x \cup y, \\ x \cdot y &:= \mathrm{cl}(\mathrm{int}(x \cap y)), \\ x^* &= \mathrm{cl}(X \setminus x). \end{aligned}$$

The contact relation is given by the standard Whiteheadean contact relation $xCy$ iff $x \cap y \neq \emptyset$.

Since their introduction several representation theorems for BCA's were proven. The most general version is the following:

**Theorem 3 (Representation Theorem [5]).** *For each Boolean contact algebra $\langle B, C\rangle$ there exists an embedding $h : B \to \mathrm{RC}(X)$ into the Boolean algebra of regular closed sets of a topological space $\langle X, \tau\rangle$ with $aCb$ iff $h(a) \cap h(b) \neq \emptyset$. $h$ is an isomorphism if $B$ is complete.*

Notice that the original theorem lists further properties of the topological space which are not important for the current work.

## 4 The Skeleton as a BCA

As already mentioned in Section 2.2 the skeleton of a Stonian p-ortholattice is a Boolean algebra. In this section we verify that it is in fact a BCA with a contact relation induced by the outer lattice.

**Theorem 4.** *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice, then $S(L)$ together with*

$$aCb \iff a \not\leq b^\perp$$

*is a Boolean contact algebra.*

*Proof.* C0. Assume $0Ca$ for an $a \in S(L)$. Then $0 \not\leq a^\perp$, a contradiction.
C1. From $a \leq a^\perp$ we conclude $a = 0$, and, hence, C1.
C2. $aCb$ implies $a \not\leq b^\perp$, which is equivalent to $b \not\leq a^\perp$. The latter shows $bCa$.
C3. Let $aCb$ and $b \leq c$. This implies $a \not\leq b^\perp$ and $c^\perp \leq b^\perp$. Together we conclude $a \not\leq c^\perp$, and , hence, $aCc$.
C4. Assume $aC(b+c)$. Then we have $a \not\leq (b+c)^\perp = b^\perp \cdot c^\perp$. This implies $a \not\leq b^\perp$ or $a \not\leq c^\perp$, and, hence, $aCb$ or $aCc$. ☐

Notice that the definition of $C$ in the theorem above uses an element $b^\perp$ that is not necessarily in the skeleton, i.e. the definition of $C$ is external to the Boolean algebra $S(L)$. By definition of the skeleton, all elements in $S(L)$ are regular closed.

**Lemma 3.** *Let $\langle L, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a Stonian p-ortholattice and $\langle S(L), C\rangle$ its skeleton BCA. Then we have:*

1. *$S(L)$ is dense in $L$ iff $C$ satisfies* C5.
2. *$L$ is $*$-normal iff $C$ satisfies* C6.
3. *$L$ is connected iff $C$ satisfies* C7.

*Proof.* 1. Assume $S(L)$ is dense in $L$. We want to show that C5' holds. Therefore, let $1 \neq x \in S(L)$. Then $x^\perp \neq 0$ which implies that there is an element $0 \neq y \in S(L)$ with $y \leq x^\perp$, i.e. $x(-C)y$. Conversely, assume C5', and let $0 \neq y \in L$. If $y^{++} = 0$ we conclude $y \leq y^{**} = (y^{\perp\perp})^{**} = y^{\perp+**} = y^{++**} = 0^{**} = 0$, a contradiction. This implies $y^* = y^{\perp**} = y^{++\perp} \neq 1$ and $y^* \in S(L)$. By C5' there is an element $0 \neq x \in S(L)$ with $y^*(-C)x$, i.e. $y^* \leq x^\perp$. The latter implies $x \leq y^{*\perp} = y^{++} \leq y$.
2. Assume $L$ is $*$-normal, and let $x, y \in S(L)$ with $x(-C)y$. Then we have $x^{**} = x \leq y^\perp = y^{**\perp} = y^+$. We obtain an element $c \in L$ with $x^{**} \leq c^{++}$ and $y^{**} \leq c^+$. The elements $x, y$ and $c^*$ are closed and we get

$$x = x^{**} \leq c^{++} = c^{*\perp} \quad \text{and} \quad y = y^{**} \leq c^+ = c^{+++} = c^{**\perp},$$

which implies $x(-C)c^*$ and $y(-C)c^{**}$ and thus C6 holds. Conversely, let $a^{**} \not\leq c^{++} = c^{*\perp}$ or $b^{**} \not\leq c^+ = c^{**\perp}$ for all $c \in L$. Then $a^{**}Cc^*$ and $b^{**}Cc^{**}$ for all $c \in L$. Since $\{c^* \mid c \in L\} = S(L)$ we conclude by C6 that $a^{**}Cb^{**}$, and, hence, $a^{**} \not\leq b^{**\perp} = b^+$.
3. Assume $C$ does not satisfy C7. Then there is a closed element $x \neq 0, 1$ with $x(-C)x^*$, i.e. $x \leq x^{*\perp} = x^{++}$. The latter shows that $x$ is also open, and, hence, $L$ is not connected.
Conversely, assume $L$ is not connected. Then there is a clopen element $x \neq 0, 1$. We conclude $x = x^{++} = x^{*\perp}$ which implies $x(-C)x^*$. □

We want to illustrate the previous lemma by some examples.

*Example 1.* Consider the Stonian p-ortholattices $C_{18}$ and $C_{14}$ from Figure 3 and 4. The pairs $(1, 0)$, $(d, c^{++})$, $(d^{++}, c)$, $(a, f^{++})$, $(a^{++}, f)$, $(e, b^{++})$, $(e^{++}, b)$ define the orthocomplements of each other. The pseudocomplements are given by $1^* = 1$, $0^* = 0$ and for all other elements $\{x, x^{++}\}^* = x$. The closed elements are $0, a, b, c, d, e, f, 1$ and the open elements are $0, a^{++}, b^{++}, c^{++}, d^{++}, e^{++}, f^{++}, 1$. Consequently, the only clopen elements are 0 and 1. On the other hand every element of the skeleton is in contact to its complement (within the skeleton). For example, we have $a^* = f$ and $f^\perp = a^{++}$. Since $a$ is not open, i.e. $a \neq a^{++}$, we obtain $a \not\leq a^{++} = f^\perp = a^{*\perp}$.

$S(L)$ is not dense in either of those Stonian p-ortholattices. For example, $a^{++} \neq 0$ but there is no closed element between $a^{++}$ and 0. As a consequence the skeleton is not extensional. We have $f \neq 1$ and no non-zero closed element is smaller than $f^\perp = a^{++}$.

Finally, both lattices are not $*$-normal. For example, the closed elements $a$ and $c$ satisfy $a = a^{**} \leq d^{++} = c^{+}$. The open elements above $a$ are $d^{++}, 1$ but none of $d^{+} = c^{++}, 1^{+} = 0$ is above $c$. Consequently, the skeleton does not satisfy C6. Indeed, $a \leq d^{++} = c^{\perp}$, and, hence, $a(-C)c$, but $C(a) = S(L) \setminus \{0, c\}$ and $C(b) = S(L) \setminus \{0, a\}$ so that we have $aCb$ or $cCb^{*}$ for all $b \in S(L)$.

*Example 2.* For the second example consider the structure $\mathrm{RT}(\mathbb{R})$ of the real line with the usual topology. Notice that, for example, the set of all rationals $r$ between 0 and 1 is not in $\mathrm{RT}(\mathbb{R})$ since $\mathrm{cl}(r) = [0, 1]$ and $\mathrm{cl}(\mathrm{int}(r)) = \mathrm{cl}(\emptyset) = \emptyset$. The skeleton of this Stonian p-ortholattice is the Boolean algebra of all regular closed sets. Since $\mathrm{RT}(\mathbb{R})$ has just two clopen elements, namely $\emptyset$ and $\mathbb{R}$, its skeleton satisfies C7. Furthermore, for every element $x \in \mathrm{RT}(\mathbb{R})$ there is a non-empty regular closed set included in $\mathrm{int}(x)$. Therefore, the skeleton is extensional. Finally, the space is normal so that any pair of disjoint regular closed sets can be separated by disjoint open sets, i.e. $\mathrm{RT}(\mathbb{R})$ is $*$-normal, and, hence, its skeleton satisfies C6.

## 5 Embedding a BCA into a Stonian p-Ortholattice

In this section we focus on the converse process. We verify that every BCA is isomorphic to the skeleton of some Stonian p-ortholattice. The following theorem shows a way how to construct the Stonian p-ortholattice.

**Theorem 5.** *Let $\langle B, C \rangle$ be an arbitrary BCA. Then there is a Stonian p-ortholattice $\langle L, +, \cdot, ^{*}, ^{\perp}, 0, 1 \rangle$ so that the skeleton $S(L)$ is isomorphic to $\langle B, C \rangle$.*

*Proof.* Let $\langle X, \tau \rangle$ be the topological space induced by Theorem 3 and let $h : B \to RC(X)$. Define $L = \{x \in \mathrm{RT}(X) \mid \exists b \in B : \mathrm{cl}(x) = h(b)\}$. Notice that if $x \in L$, i.e. $\mathrm{cl}(x) = h(b)$ for some $b \in B$, then we have

$$\begin{aligned}
h(b^{*}) &= \mathrm{cl}(X \setminus h(b)) && h \text{ homomorphism} \\
&= \mathrm{cl}(X \setminus \mathrm{cl}(x)) \\
&= X \setminus \mathrm{int}(\mathrm{cl}(x)) \\
&= X \setminus \mathrm{int}(x) && x \in \mathrm{RT}(X) \\
&= \mathrm{cl}(X \setminus x).
\end{aligned}$$

We have to show that the skeleton $S(L)$ is exactly the image of $h$, that $L$ is closed with respect to all operations of $\mathrm{RT}(X)$, and that $aCb$ iff $h(a) \nsubseteq X \setminus h(b)$ for all $a, b \in B$.

Obviously every $h(a)$ is closed, i.e. $h(a) \in S(L)$. Conversely, suppose $x$ is closed. Then $x = \mathrm{cl}(x) = h(b)$ for some $b \in B$, i.e. $x$ is in the image of $h$.

Now, suppose there are elements $b_1, b_2 \in B$ with $\mathrm{cl}(x) = h(b_1)$ and $\mathrm{cl}(y) = h(b_2)$. and consider the following computations:

$$
\begin{aligned}
\mathrm{cl}(x \cap^* y) &= \mathrm{cl}(\mathrm{int}(x \cap^* y)) && x \cap^* y \in \mathrm{RT}(X) \\
&= \mathrm{cl}(\mathrm{int}(x) \cap \mathrm{int}(y)) && \text{Lemma 2(2)} \\
&= \mathrm{cl}((X \setminus \mathrm{cl}(X \setminus x)) \cap (X \setminus \mathrm{cl}(X \setminus y))) && \\
&= \mathrm{cl}((X \setminus h(b_1^*)) \cap (X \setminus h(b_2^*))) && \text{see above} \\
&= \mathrm{cl}(X \setminus (h(b_1^*) \cup h(b_2^*))) && \\
&= \mathrm{cl}(X \setminus h(b_1^* + b_2^*)) && h \text{ homomorphism}
\end{aligned}
$$

$$
\begin{aligned}
&= h((b_1^* + b_2^*)^*) && h \text{ homomorphism} \\
&= h(b_1 \cdot b_2), && \\
\mathrm{cl}(x \cup^* y) &= \mathrm{cl}(x) \cup \mathrm{cl}(y) && \text{Lemma 2(1)} \\
&= h(b_1) \cup h(b_2) && \\
&= h(b_1 + b_2), && h \text{ homomorphism} \\
\mathrm{cl}(x^*) &= \mathrm{cl}(\mathrm{cl}(X \setminus x)) && \\
&= \mathrm{cl}(X \setminus x) && \\
&= h(b_1^*), && \text{see above} \\
\mathrm{cl}(x^\perp) &= \mathrm{cl}(X \setminus x) && \\
&= h(b_1^*). && \text{see above}
\end{aligned}
$$

Finally, using Theorem 3 we immediately conclude $aCb$ iff $h(a) \cap h(b) \neq \emptyset$ iff $h(a) \nsubseteq X \setminus h(b)$. □

The Stonian p-ortholattice from the previous theorem is not necessarily the only lattice that has $\langle B, C \rangle$ as its skeleton BCA.

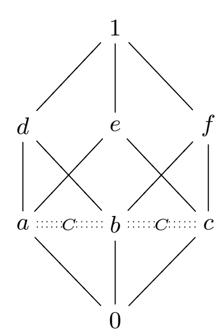


**Fig. 2.** A Boolean Contact Algebra

*Example 3.* Consider the BCA from Figure 2. The diagram just shows external connection between atoms ($C$ edges). The actual contact relation $C$ on this Boolean algebra is given as the smallest relation that contains those edges, overlap and is upwards closed, i.e. closed with respect to C3. Notice that this BCA satisfies C7 but neither C5 nor C6.

The topological space that is constructed in the proof of Theorem 3 (see [5]) for this example is based on a set isomorphic to $X = \{0, 1, 2, 3, 4\}$ with open sets

$$\tau = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}, \{0,1,3\},$$
$$\{1,2,4\}, \{0,1,2,3\}, \{0,1,2,4\}, \{0,1,2,3,4\}\}.$$

From those open sets $\{0, 1\}, \{1, 2\}, \{0, 1, 2\}, \{0, 1, 2, 4\}$ and $\{0, 1, 2, 3\}$ are not regular open, e.g. we have $\mathrm{int}(\mathrm{cl}(\{1, 2\})) = \mathrm{int}(\{1, 2, 3, 4\}) = \{1, 2, 4\}$. We obtain the Stonian p-ortholattice from Figure 3. Notice that this is the lattice $C_{18}$, one of the four structures characterizing models of $RT$ [10].

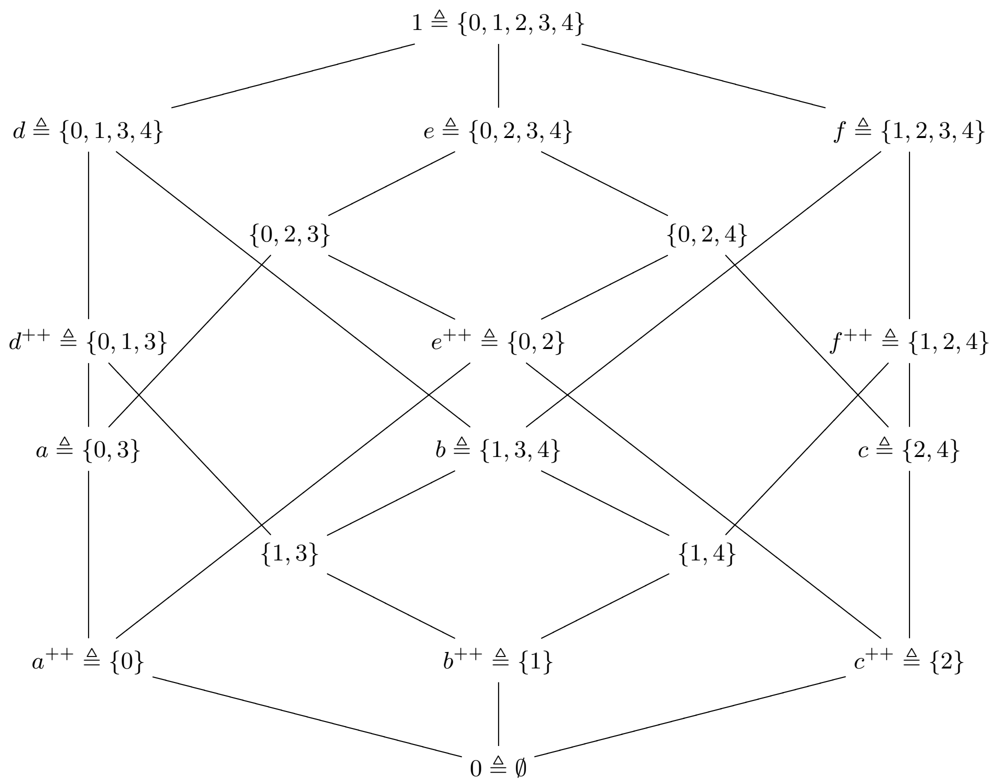


**Fig. 3.** The Stonian p-Ortholattice $C_{18}$

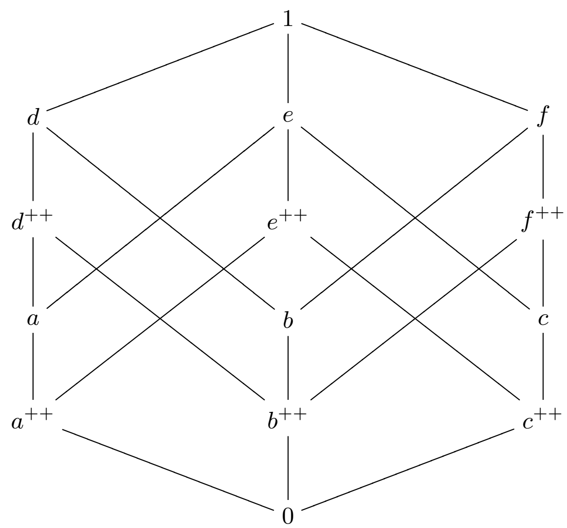


**Fig. 4.** The Stonian p-Ortholattice $C_{14}$

On the other hand, the Stonian p-ortholattice $C_{14}$ from Figure 4 has the same skeleton as $C_{18}$. A careful investigation also shows that the contact relations induced on the skeleton is the same for both lattices.

## 6 Conclusion and Future Work

In this paper we have established the relationship between BCAs and Stonian p-ortholattices. Due to the equivalence of those theories to subtheories of RCC and $RT_0$ we obtain similar results for those mereotopologies. Our theoretical work directly implies that every connected, $*$-normal model of $RT^-$ with a dense skeleton is a model of the full RCC. On the other extreme, any model of the $RT^-$ is a model of the RCC without the axioms C5, C6, and C7. Conversely, every model of the RCC is a model of $RT^-$: The BCA corresponding to an RCC model is isomorphic to the skeleton $S(L)$ of some Stonian p-ortholattice $L$ by Theorem 5. However, the skeleton $S(L)$ itself is a Stonian p-ortholattice, since the Boolean algebras are a subclass of the Stonian p-ortholattices. Consequently, every RCC model is a $RT^-$ model as well. With little effort we can show the relation to models of full $RT_0$: if the RCC model contains some minimal set of regular open sets, it can always be extended to a model of the full theory $RT_0$.

More generally speaking, by using previously published algebraic representations of the theories RCC and $RT_0$ and clarifying the relationship between their algebraic representations, this work contributes to the understanding of the relationship between different logical theories of mereotopology. Establishing a formal relationship between models of subtheories of RCC and $RT_0$ would have been extremely difficult without the lattice-theoretic account of their models. This emphasizes the benefit of algebraic representations of logical theories, in particular of mereotopological theories. Ultimately, we want to gain a deeper understanding of the relationship between the major theories of mereotopology. Part of our future work will focus on algebraic representations of other mereotopologies. In the long-term, this will allow to obtain similar relationships between the various mereotopological theories. By doing so, we hope to foster a deeper understanding of the different mereotopologies, their models, and the relationships amongst them.

As a separate issue, even though we verified that the structure $\mathrm{RT}(X)$ for a topological space $X$ is indeed a Stonian p-ortholattice, a topological representation theorem has not yet been established. Future work will concentrate on this aspect as well. In particular, it is of interest whether a representation theorem can be developed that corresponds on the skeleton to the known results for BCAs.

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