A Naïve Theory of Dimension for Qualitative Spatial Relations

Torsten Hahmann
Department of Computer Science
University of Toronto
torsten@cs.toronto.edu

Michael Grüninger
Department of Mechanical and Industrial Engineering
University of Toronto
gruninger@mie.utoronto.ca

Abstract

We present an ontology consisting of a theory of spatial dimension and a theory of dimension-independent mereological and topological relations in space. Though both are fairly weak axiomatizations, their interplay suffices to define various mereotopological relations and to make any necessary dimension constraints explicit. We show that models of the INCH Calculus and the Region-Connection Calculus (RCC) can be obtained from extensions of the proposed ontology.

1 Introduction

It has long been argued that commonsensical theories of space must include qualitative spatial descriptions using topological, e.g. contact, and mereological, e.g. parthood, relations. Actual theories of space axiomatizing these topological and mereological relations, however, have often been restricted to one class of 'foundational' entities of uniform dimension: usually either points or regions.

A dimension-independent approach should favor neither the bottom-up approach (defining higher-dimensional entities in terms of points) of classical geometry nor the top-down approach (taking higher-dimensional regions as foundational and reconstructing dependent lower-dimensional entities) employed in standard mereotopology. Instead, our objective is to axiomatize topological and mereological relations in a theory where entities of various dimensions coexist as first-class domain objects, e.g. for qualitatively describing sketch maps (Fig. 1). The challenge is to separate the relations that can hold between entities regardless of their dimension from the relations that constrain the dimensions of the involved entities. Needless to say that these relations shall align with common intuitions of space. The need for such a dimension-independent theory of space has been reiterated recently by (Frank 2010). Here, we axiomatize such a theory from first principles in a modular matter. The following guidelines frame our endeavour:

1. Separating the notion of dimension from an axiomatization of dimension-independent spatial relations;
2. Expressing dimension constraints (if any exist) explicitly for the defined mereotopological relations;

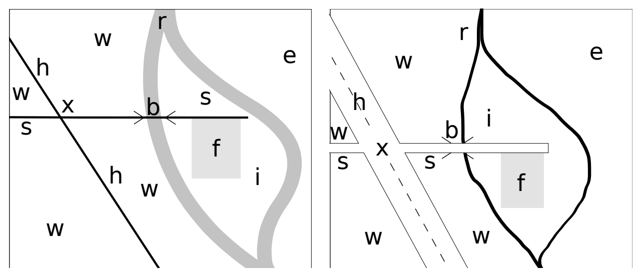

Figure 1: Two sketches of a map depicting an island (i) in a river (r) and a street (s) across a bridge (b) over a river arm to the island. The street intersects (x) a highway (h). There are regions east (e) and west (w) of the river. Roads as well as rivers can be treated as (complex) lines or as regions.

3. Identifying metalogical relationships between the modules of the axiomatization (definitional, conservative, and non-conservative extensions; cf. Grüninger *et al.* 2010);
4. Verifying the consistency and basic properties of the spatial relations automatically.

We show that many commonsensical refinements of contact are definable in our ontology consisting of a theory of dimension and a theory of dimension-independent mereotopology based on a single mereological or topological primitive ('containment' or 'contact'). Subsequently, we will show how models of two mereotopologies can be obtained from extensions to our proposed ontology.

1.1 Background and related work

A wealth of theories capturing the mereotopological relations between equi-dimensional spatial regions, such as surfaces in a 2D space or intervals in a 1D space, are known but few capture the relations between regions of mixed dimensions. Rare exceptions focus on capturing boundaries as lower-dimensional entities (Galton 1996; 2004), or on area-line and line-line relations in two-dimensional space, e.g. (Egenhofer & Herring 1991; Clementini, Di Felice, & Oosterom 1993). (Galton 2004) contains a more comprehensive and still up-to-date survey of dimension in spatial relations.

Two main approaches towards a theory of topological and mereological relations are known, namely the logical approach (e.g. the RCC: Cohn *et al.* 1997) and the intersection approach (Egenhofer 1991; Egenhofer & Herring 1991) classifying topological relations by the intersection of interiors, boundaries and exteriors. Egenhofer & Herring (1991), Clementini, Di Felice, & Oosterom (1993), and McKenney *et al.* (2005) refined those to a total of 68, 52, and 128, respectively, topological relations that account for the dimension of the overlap/contact between any combinations of simple points, lines, and areas. Amongst these relations, the authors of those three proposals distinguish alone 33 (18, 61) relations amongst simple lines in a two-dimensional space. While these approaches are precise in distinguishing a variety of mereotopological configurations, humans are incapable of dealing with such a large set of relations. Moreover, these relations only apply to spatial entities in 2D space, i.e. only points, lines and surfaces are considered. The alternative calculus-based approach in Clementini *et al.* (1993) on the other hand deals with a small set of commonsensical relations, but is very ad-hoc and relies on an involved topological apparatus. Moreover, neither of those approaches enables us to explicitly reason about dimension.

With respect to the calculus-based (or 'logical') approaches, our work is most closely related to the INCH Calculus (Gotts 1996) which uses a primitive $INCH(x,y)$ with the intended meaning 'x includes a chunk of y' (a chunk being an equi-dimensional part). This theory is capable of describing strong contact (what we call partial overlap and incidence) but is incapable of capturing superficial contact, e.g. when two equi-dimensional regions share a boundary or touch in a point. Nevertheless our axiomatization has been inspired by that of (Gotts 1996) – we generalize and organize it into a modular set of ontologies. However, we start with a *dimension-independent* mereological primitive, containment, and the definition of its *dimension-independent* topological counterpart, contact, and then show how other *dimension-dependent* mereotopological predicates, such as *INCH*, are definable. Thereby, we make the dimension constraints of various mereotopological relations explicit. In particular, we include the important relations 'superficial contact' and 'incidence'.

Other related calculus-based approaches include the extension of classical mereotopology with dimensions by (Galton 1996) and (Galton 2004). Both are chiefly concerned with the definability of dependent lower-dimensional entities, in particular boundaries, in a top-down approach to mereotopology. Galton (1996) shows how such lower-dimensional entities can be accommodated through a set of equivalence classes that only contain entities of the same dimension while restricting parthood to members of the same equivalence class. Similarly, (Galton 2004) defines dependent lower-dimensional entities as regular closed subsets of the boundaries of higher-dimensional entities.

1.2 Preliminaries

Our axiomatization uses standard unsorted first-order logic with equality where $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ denote the connectives of negation, and, or, implication, and 'if and only if', respectively. Variables appearing free in sentences are assumed to be universally quantified unless otherwise stated. All variables range over 'spatial entities', also called 'regions' and referred to as 'extents' by (Gotts 1996). We assume each spatial entity can be assigned a unique dimension which is uniform across all parts[1]. In particular, a region and the set of all points contained in the region are different.

Sentences are labelled according to the schema '[theory]-[type][number]' (e.g. P-T1) where the first letter(s) indicate the theory (e.g. P=parthood, CD=containment & dimension), while the type distinguishes axioms (A), definitions (D), theorems (T), extensions (E), and mappings (M).

The consistency of the various theories has been checked with Mace4 and all theorems have been verified using the automated theorem prover (ATP) Prover9 (McCune 2010)[2].

2 A naïve theory of dimension

Various notions of dimension have been employed within theories of qualitative space. We want to axiomatize dimension in the weakest possible way which is still suitable for defining spatial relations that are limited to entities of certain (relative) dimensions. For example we want to be able to express that region A has a higher dimension than region B or the intersection of regions A and B has a lower dimension than either one. Thereby it is unnecessarily restrictive to e.g. require that dimensions can be added or subtracted or restrict the total number of distinct dimensions. In other words, the sought axiomatization should be just strong enough to allow us to compare the dimensions of spatial entities.

A brief look at the various definitions of dimension in topology can be of help. There we find the small and large inductive dimensions, the Lebesgue covering dimension (cf. McKenney *et al.* 2005), the Haussdorff dimension, and the notion of dimension in the theory of manifolds. (Engelking 1995) gives a good overview of dimension from the topological perspective. Other notions of dimensions, e.g. those used for vector spaces or Hilbert spaces, are difficult to include in a qualitative theory of space.

A theory of dimension that suits our needs can be constructed reusing core ideas from inductive definitions of dimension. However, the relevant topological definitions are either still overly restrictive or rely on a heavy topological apparatus of which we would like to rid ourselves. We construct our basic theory of dimension, T_{dim}, as weak axiomatizations of the binary primitive relations $<$ and $=_{dim}$. Their intended interpretations are 'x has a lower dimension than y' ($x < y$) and 'x and y have the same dimension' ($x =_{dim} y$). $<$ is irreflexive, asymmetric, and transitive (strict partial order). $=_{dim}$ meanwhile is reflexive, symmetric, and transitive (equivalence relation). Though in many settings one would define $=_{dim}$ in terms of $<$ as $x =_{dim} y \leftrightarrow x \not< y \wedge y \not< x$, this is not required in T_{dim}. We thereby permit incomplete information regarding the dimension of entities, i.e. models in which neither $x =_{dim} y$, nor, $x < y$, nor $y < x$ hold. D-A7 to D-A9 govern the relationship between $<$ and $=_{dim}$ while D-A11, D-A12 ensure that a potential zero region (we use *ZEX* from (Gotts 1996)) is unique and of lowest dimension. D-A10 demands a lowest-dimensional entity (apart from *ZEX*) with-

[1]Containment is the dimension-independent version of 'parthood' that can express that, e.g., an object contains a crack.

[2]Available at `www.cs.toronto.edu/~torsten/DCT`

out preventing infinite-dimensional models. As this paper will show, this theory is sufficiently strong to distinguish mereotopological relations that depend on dimensions.

(D-A1) $x \not< x$ ($<$ irreflexive)
(D-A2) $x < y \rightarrow y \not< x$ ($<$ asymmetric)
(D-A3) $x < y \wedge y < z \rightarrow x < z$ ($<$ transitive)
(D-A4) $x =_{dim} x$ ($=_{dim}$ reflexive)
(D-A5) $x =_{dim} y \rightarrow y =_{dim} x$ ($=_{dim}$ symmetric)
(D-A6) $x =_{dim} y \wedge y =_{dim} z \rightarrow x =_{dim} z$ ($=_{dim}$ transitive)
(D-A7) $x < y \rightarrow \neg x =_{dim} y$ ($=_{dim}$ and $<$ incompatible)
(D-A8) $x =_{dim} y \wedge z < x \rightarrow z < y$ ($=_{dim}$ renders $<$ transitive)
(D-A9) $x =_{dim} y \wedge x < z \rightarrow y < z$ ($=_{dim}$ renders $<$ transitive)
(D-A10) $\exists x(\neg ZEX(x) \wedge \forall y(y < x \rightarrow ZEX(y)))$ (lowest dim.)
(D-A11) $ZEX(x) \wedge ZEX(y) \rightarrow x = y$ (unique ZEX)
(D-A12) $ZEX(x) \wedge \neg ZEX(y) \rightarrow x < y$ (ZEX has minimal dim.)

While T_{dim} = {D-A1–D-A12} is agnostic about the existence of a zero region to accommodate extensions in which such region is either desirable or convenient, its extensions by Z-A1 to T^0_{dim} and by NZ-A1 force/prevent a zero region.

(Z-A1) $\exists x ZEX(x)$ (existence of a ZEX)
(NZ-A1) $\neg ZEX(x)$ (no ZEX exists)

Now we introduce additional useful definitions for dimension. The module $T_{dim-defs} = T_{dim} \cup$ {D-D1–D-D6} is a (conservative) definitional extension of T_{dim}. D-D6 requires some explanation: it makes the distinction between $x =_{dim} y$ and $x \not< y \wedge y \not< x$ explicit. We allow models in which dimensionally incomparable regions have 'possibly the same dimension'. This can be useful when the dimension of an entity differs depending on the relation to other entities. A building, e.g., can be 2D in relation to the street it is located on, but 3D when related to its floors which themselves might be 2D on a floor plan, but 3D when talking about storage space. Entities of minimal or maximal dimension as defined by D-D4 and D-D5 are thus not necessarily of minimum or maximum dimension because e.g. two maximal-dimensional entities might be incomparable with respect to their dimensions. In $T_{dim-defs}$ basic properties of $\not\gtrless$ and of its interaction with $<$ and $=_{dim}$ are provable (D-T1 to D-T5).

(D-D1) $x > y \leftrightarrow x < y$ (greater dim.)
(D-D2) $x \leq y \leftrightarrow x < y \vee x =_{dim} y$ (lesser or equal dim.)
(D-D3) $x \geq y \leftrightarrow x > y \vee x =_{dim} y$ (greater or equal dim.)
(D-D4) $MaxDim(x) \leftrightarrow \forall y(x \not< y)$ (maximal-dimensional entity)
(D-D5) $Atom(x) \leftrightarrow \neg ZEX(x) \wedge \forall y(y < x \rightarrow ZEX(y))$
(atom = minimal-dimensional entity apart from ZEX)
(D-D6) $x \not\gtrless y \leftrightarrow x \not< y \wedge y \not< x$ (possibly equal dimension)
(D-T1) $x \not\gtrless x$ ($\not\gtrless$ reflexive)
(D-T2) $x \not\gtrless y \rightarrow y \not\gtrless x$ ($\not\gtrless$ symmetric)
(D-T3) $x =_{dim} y \wedge y \not\gtrless z \rightarrow x \not\gtrless z$ ($\not\gtrless$ transitive under $=_{dim}$)
(D-T4) $x =_{dim} y \rightarrow x \not\gtrless y$ ($=_{dim}$ implies $\not\gtrless$)
(D-T5) $x < y \rightarrow x \neq_{dim} z \vee y \neq_{dim} z$

Two further extensions are of practical importance. First, the theory of bounded dimension $T_{dim-bounded}$, which non-conservatively extends T_{dim} by D-E1 to D-E3 ensuring that unique minimum and maximum dimensions exist while coincidence of maximum and minimum dimension forces equi-dimensionality of a model. Independently thereof, we can require regions to be dimensionally comparable (D-E4). Such theory of linear dimension $T_{dim-linear}$ extends T_{dim} non-conservatively; it entails $x =_{dim} y \leftrightarrow x \not\gtrless y$ and D-E3.

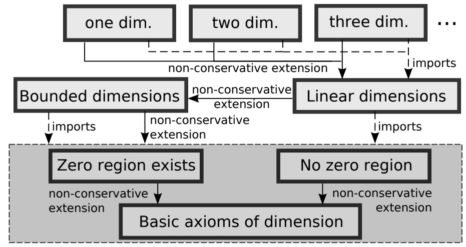

Figure 2: Modules of the theory of dimension.

(D-E1) $\exists x(Atom(x) \wedge \forall y(Atom(y) \rightarrow x =_{dim} y))$
(D-E2) $\exists x(MaxDim(x) \wedge \forall y(MaxDim(y) \leftrightarrow x =_{dim} y))$
(unique minimum and maximum dimensions)
(D-E3) $\neg ZEX(x) \wedge MaxDim(y) \wedge Atom(y) \rightarrow y =_{dim} x$
(equi-dimensionality if maximum = minimum dimension)
(D-E4) $x < y \vee y < x \vee x =_{dim} y$ (comparability required)

Further extensions can limit the number of distinct dimensions to any other particular number (cf. Fig. 2) as necessary for e.g. k-partite incidence geometry.

3 Dimension-independent spatial relations

We proceed by examining the mereological and topological relations that can hold between spatial entities independent of their dimension. On the mereological side this is spatial containment, denoted by $Cont(x,y)$, and on the topological side it is contact, denoted by $C(x,y)$. Though we choose $Cont$ as spatial primitive, C would serve equally well as primitive which can define $Cont$. This interchangeability of a topological and mereological primitive has first been observed in (Hahmann, Winter, & Grüninger 2009) for the equi-dimensional mereotopology of (Asher & Vieu 1995).

3.1 Containment as mereological relation

What parthood is to equi-dimensional mereotopology, containment is to dimension-independent mereotopology. In its point-set interpretation, we say 'y contains x', i.e. $Cont(x,y)$, if every point in space occupied by x is also occupied by y. A region can contain not only a (smaller) region of the same dimension (equi-dimensional parthood), but also a lower-dimensional entity. E.g. a 2D-surface can contain another 2D-surface, a line, or a point. Containment is a non-strict partial order. We again use ZEX to denote a zero region which neither contains nor is contained in any other region. For the basic theory of containment, T_{cont} = {C-A1–C-A4} we make no assumption about the (non-)existence of a zero region. Two extensions are feasible: $T^0_{cont} = T_{cont} \cup$ {Z-A1} and $T^{\neg 0}_{cont} = T_{cont} \cup$ {NZ-A1}.

(C-A1) $\neg ZEX(x) \rightarrow Cont(x,x)$ ($Cont$ reflexive)
(C-A2) $Cont(x,y) \wedge Cont(y,x) \rightarrow x = y$ ($Cont$ antisymmetric)
(C-A3) $Cont(x,y) \wedge Cont(y,z) \rightarrow Cont(x,z)$ ($Cont$ transitive)
(C-A4) $ZEX(x) \rightarrow \forall y(\neg Cont(x,y) \wedge \neg Cont(y,x))$
(no entity contains or is contained in ZEX)

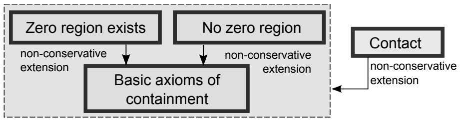

Figure 3: Modules of the theory of containment.

3.2 Contact as a definable topological relation

Now contact C is definable in terms of containment (C-D). Those familiar with equi-dimensional mereotopology might notice the close resemblance to the common mereotopological definition of overlap as $O(x,y) \leftrightarrow \exists z(P(z,x) \wedge P(z,y))$. We can subsequently prove that C is reflexive and symmetric and does not hold for the zero-region. We can also show monotonicity of C (C-T4) in case of containment. The inverse of C-T4 is not entailed and thus posited as axiom C-A5. Hence $T_{cont-c} = T_{cont} \cup$ {C-D, C-A5} is a non-conservative extension of T_{cont}. C-T4 and C-A5 imply that *Cont* is also definable by contact C as primitive relation.

(C-D) $C(x,y) \leftrightarrow \exists z(Cont(z,x) \wedge Cont(z,y))$ (contact)

(C-T1) $\neg ZEX(x) \rightarrow C(x,x)$ (C reflexive)

(C-T2) $C(x,y) \rightarrow C(y,x)$ (C symmetric)

(C-T3) $ZEX(x) \rightarrow \forall y(\neg C(x,y))$ (nothing in contact with ZEX)

(C-T4) $Cont(x,y) \rightarrow \forall z(C(z,x) \rightarrow C(z,y))$ (*Cont* implies C monotone)

(C-A5) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \forall z(C(z,x) \rightarrow C(z,y)) \rightarrow Cont(x,y)$ (C monotone implies *Cont*)

As side-effect, C-D forces a mereological closure. Contact between two entities requires the existence of a common contained entity – interpretable as intersection. A useful and common assumption is extensionality of C.

(C-E1) $x = y \leftrightarrow \forall z(C(z,x) \leftrightarrow C(z,y))$ (C extensional)

4 Interaction of dimension and containment

The basic theory of containment, T_{cont}, can be combined with T_{dim} by axiomatizing the direct relation between containment and dimension: if x is contained in y, then x must have a dimension that is the same as or lower than that of y. We obtain $T_{cont-dim} = T_{cont-c} \cup T_{dim} \cup$ {CD-A1}.

(CD-A1) $Cont(x,y) \rightarrow x < y \vee x =_{dim} y$

Surprisingly, all theorems in this section, with the exception of exhaustiveness of the contact relations (CD-T4), can be proved without using CD-A1.

Another common assumption is indivisibility of atoms (cf. CD-E1), the non-zero entities of lowest dimension. Indivisibility is justified as long as atoms represent points. If the entities of the lowest dimension represent lines or surfaces it is too strong an assumption. Though a necessary extension e.g. for incidence geometry, we do not include CD-E1 in our general theory of containment and dimension.

(CD-E1) $Atom(x) \rightarrow \forall y(Cont(y,x) \rightarrow x = y)$ (indivisible atoms)

We use $T_{cont-dim}$ to define three types of contact that depend on the dimension of the entities in contact and/or the dimension of the common entity. We distinguish two types of strong contact, namely partial overlap and incidence, and weak (or superficial) contact. We show that the three form an exhaustive and pairwise disjoint set of contact relations.

First, notice that the traditional notion of parthood, i.e. containment between two equi-dimensional entities, is easily defined in $T_{cont-dim}$. We verify that parthood is a non-strict partial order (P-T1 to P-T3) which implies contact (P-T8) and we prove simple transitivity properties in interaction with dimension constraints (P-T4 to P-T7).

(P-D) $P(x,y) \leftrightarrow Cont(x,y) \wedge x =_{dim} y$ (parthood)

(P-T1) $\neg ZEX(x) \rightarrow P(x,x)$ (P reflexive)

(P-T2) $P(x,y) \wedge P(y,x) \rightarrow x = y$ (P antisymmetric)

(P-T3) $P(x,y) \wedge P(y,z) \rightarrow P(x,z)$ (P transitive)

(P-T4) $P(x,y) \wedge z < x \rightarrow z < y$

(P-T5) $P(x,y) \wedge y < z \rightarrow x < z$

(P-T6) $P(x,y) \wedge z =_{dim} x \rightarrow z =_{dim} y$

(P-T7) $P(x,y) \wedge z =_{dim} y \rightarrow x =_{dim} z$

(P-T8) $P(x,y) \rightarrow C(x,y)$ (parthood requires contact)

4.1 Strong contact

Equi-dimensional: Partial overlap Parthood allows us to define partial overlap as the strongest contact holding when two regions share a part. Partial overlap is a reflexive and symmetric relation requiring equi-dimensionality.

(PO-D) $PO(x,y) \leftrightarrow \exists z(P(z,x) \wedge P(z,y))$ (partial overlap)

(PO-T1) $\neg ZEX(x) \rightarrow PO(x,x)$ (PO reflexive)

(PO-T2) $PO(x,y) \rightarrow PO(x,y)$ (PO symmetric)

(PO-T3) $PO(x,y) \rightarrow x =_{dim} y$ (PO requires equi-dimensionality)

Non-equi-dimensional: Incidence Two entities of different dimension can also be in strong contact. We generalize partial overlap to incidence by requiring that they share a region that is part of one (instead of both) of them, but do not share a region that is part of both. The theorems INC-T1 to INC-T6 become provable.

(INC-D) $Inc(x,y) \leftrightarrow \exists z[Cont(z,x) \wedge P(z,y) \wedge z < x] \vee \exists z[P(z,x) \wedge Cont(z,y) \wedge z < y]$ (incidence)

(INC-T1) $\neg Inc(x,x)$ (Inc irreflexive)

(INC-T2) $Inc(x,y) \rightarrow Inc(y,x)$ (Inc symmetric)

(INC-T3) $x =_{dim} y \rightarrow \neg Inc(x,y)$ (equi-dimensionality prevents incidence)

(INC-T4) $Inc(x,y) \rightarrow x < y \vee y < x$ (incidence requires comparability of entities)

(INC-T5) $Cont(x,y) \wedge x < y \rightarrow Inc(x,y)$ (containment of a lower-dimensional entity requires incidence)

(INC-T6) $Inc(x,y) \wedge P(y,z) \rightarrow Inc(x,z)$ (incidence transitive with respect to parthood)

Boundaries (and parts thereof) are special kinds of entities incident with the bounded entity. A naïve definition (INC-E1) defining boundary parts as lower-dimensional entities that arise when two non-overlapping, non-incident entities meet is far from ideal: e.g. the intersection point of two crossing lines is part of the boundary of either line. We can prevent this by excluding from the boundary those 'parts of the boundary' (BP) that are contained in two non-overlapping parts of the bounded entity. We refrain from exploring these issues further and refer to the abundant literature on boundaries, e.g. (Fleck 1996; Smith & Varzi 1997).

(INC-E1) $BP(x,y) \leftrightarrow Cont(x,y) \wedge Inc(x,y) \wedge \exists z[Cont(x,z) \wedge x < z \wedge \neg PO(z,y) \wedge \neg Inc(z,y)]$ (x is part of the boundary of y)

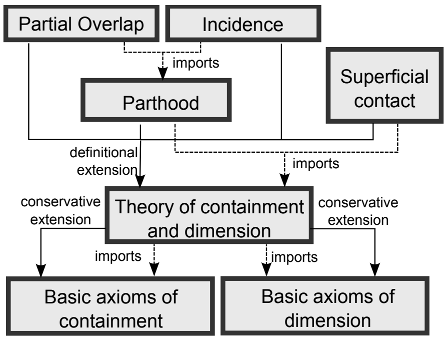

Figure 4: The mereotopological concepts parthood (and partial overlap), incidence, and superficial contact are definitional extensions of the theories $T_{cont-dim}$ and $T^0_{cont-dim}$ which combine containment with dimension.

4.2 Weak contact: Superficial contact

While partial overlap and incidence are strong contacts in the sense that the shared entity is at least an equi-dimensional part of one region, superficial contact is a weak contact. We say two entities are in superficial contact, *SC*, if the shared spatial entity has a dimension lower than both entities in contact. SC-T1 proves that this is equivalent to saying that two entities are in superficial contact if they are in contact but the shared entity is not a part of either. Superficial contact is an irreflexive and symmetric relation.

(SC-D) $SC(x,y) \leftrightarrow \exists z[Cont(z,x) \wedge Cont(z,y)] \wedge \neg\exists z[Cont(z,x) \wedge Cont(z,y) \wedge (x =_{dim} z \vee y =_{dim} z)]$ (superficial contact)

(SC-T1) $SC(x,y) \leftrightarrow C(x,y) \wedge \neg\exists z[(Cont(z,x) \wedge P(z,y)] \wedge \neg\exists z[(P(z,x) \wedge Cont(z,y)]$

(SC-T2) $\neg SC(x,x)$ (*SC* irreflexive)

(SC-T3) $SC(x,y) \rightarrow SC(y,x)$ (*SC* symmetric)

(SC-T4) $SC(x,y) \rightarrow \exists z(z < x \wedge z < y \wedge Cont(z,x) \wedge Cont(z,y))$ (*SC* requires a shared entity of a lower dimension)

Superficial contact is dimension-independent. An equi-dimensional version can only be defined if at least one region has codimension 0 such as in traditional equi-dimensional mereotopology where all regions are of the same dimension.

(SC-E1) $SC(x,y) \wedge MaxDim(x) \wedge MaxDim(y) \rightarrow EC(x,y)$ (external connection)

4.3 Exhaustiveness and disjointness

In the theory $T_{full} = T_{cont-dim} \cup$ {P-D,PO-D,Inc-D,SC-D}, *PO*, *SC*, and *Inc* are pairwise disjoint (CD-T5 to CD-T10) and are an exhaustive set of contact relations (CD-T1 to CD-T4).

(CD-T1) $PO(x,y) \rightarrow C(x,y)$

(CD-T2) $SC(x,y) \rightarrow C(x,y)$

(CD-T3) $Inc(x,y) \rightarrow C(x,y)$

(CD-T4) $C(x,y) \rightarrow PO(x,y) \vee SC(x,y) \vee Inc(x,y)$

(CD-T5) $PO(x,y) \rightarrow \neg SC(x,y)$

(CD-T6) $PO(x,y) \rightarrow \neg Inc(x,y)$

(CD-T7) $SC(x,y) \rightarrow \neg PO(x,y)$

(CD-T8) $SC(x,y) \rightarrow \neg Inc(x,y)$

(CD-T9) $Inc(x,y) \rightarrow \neg PO(x,y)$

(CD-T10) $Inc(x,y) \rightarrow \neg SC(x,y)$

5 Relationship to other spatial theories

Now we show how to extend the ontology T_{full} so that the models of the extension are models of (a) the INCH Calculus (Gotts 1996) or (b) the RCC (Cohn *et al.* 1997). More precisely, the INCH Calculus and the RCC *faithfully* and *definably interpret* (Grüninger *et al.* 2010), respectively, extensions of $T^0_{full-linear} = T_{full} \cup T^0_{dim} \cup T_{dim-linear}$. Recall that this extended theory requires a zero entity and dimensional comparability of all entities. Our extensions explicitly formalize previously implicit dimension constraints for various spatial relations such as *external contact* (EC), *includes an equi-dimensional chunk of* (INCH), or *lower-dimensional element* (EL).

5.1 INCH Calculus

We first define the intended (or 'primary') interpretation, 'x includes a chunk (a part) of y', of the relation $INCH(x,y)$ in terms of dimension and containment (I-D). The axioms (I-PA3)–(I-PA7) of the INCH Calculus are immediately provable in the presence of the definitions I-D1 to I-D5. Moreover, the defined relations *ZEX*, *GED*, *ED*, and *GD* in (Gotts 1996) correspond to our dimension predicates *ZEX*, $\geq$, $=_{dim}$, and $>$, respectively (I-D6 to I-D9). We define $T_{inch-basic} = T^0_{full-linear} \cup$ {I-D, I-D1 – I-D9}.

(I-D) $INCH(x,y) \leftrightarrow \exists z(Cont(z,x) \wedge P(z,y))$ (includes a chunk)

(I-D1) $CS(x,y) \leftrightarrow \forall z(INCH(x,z) \rightarrow INCH(y,z))$ ('x is a constituent of y')

(I-D2) $OV(x,y) \leftrightarrow INCH(x,y) \wedge INCH(y,x)$ (overlap)

(I-D3) $CO(x,y) \leftrightarrow \exists z(\neg ZEX(z) \wedge CS(z,x) \wedge CS(z,y))$ (connection)

(I-D4) $CH(x,y) \leftrightarrow INCH(x,y) \wedge \forall z(OV(x,z) \rightarrow OV(y,z))$ ('x is a chunk (equi-dimensional part) of y')

(I-D5) $EL(x,y) \leftrightarrow CS(x,y) \wedge \neg INCH(x,y)$ ('x is a (lower-dimensional) element of y')

(I-D6) $ZEX(x,y) \leftrightarrow \neg INCH(x,y)$

(I-D7) $GED(x,y) \leftrightarrow y < x \vee x =_{dim} y$ (greater or equal dim.)

(I-D8) $ED(x,y) \leftrightarrow x =_{dim} y$ (equal dimension)

(I-D9) $GD(x,y) \leftrightarrow y < x$ (greater dimension)

(I-PA3) $INCH(x,y) \rightarrow INCH(x,x)$

(I-PA4) $GED(x,y) \vee GED(y,x)$

(I-PA5) $GED(x,y) \wedge GED(y,z) \rightarrow GED(x,z)$

(I-PA6) $INCH(x,y) \wedge INCH(y,z) \wedge INCH(z,x) \rightarrow INCH(y,x)$

(I-PA7) $INCH(x,y) \rightarrow \exists z(CS(z,x) \wedge OV(z,y))$

Next, we verify that in $T_{inch-basic}$, *PO* is equivalent to *OV* and under the premise of $y < x$, *Inc* is equivalent to *INCH*. Moreover, *Cont*, *C*, *P*, and *SC* are interpretable using the defined relations *CS*, *CO*, and *CH* from the INCH Calculus.

(I-M1) $Cont(x,y) \vee ZEX(x) \rightarrow CS(x,y)$

(I-M2) $OV(x,y) \leftrightarrow PO(x,y)$

(I-M3) $CO(x,y) \rightarrow C(x,y)$

(I-M4) $P(x,y) \rightarrow CH(x,y)$

(I-M5) $(Cont(x,y) \wedge x < y) \vee ZEX(x) \rightarrow EL(x,y)$

(I-M6) $y < x \rightarrow (Inc(x,y) \leftrightarrow INCH(x,y))$

(I-M7) $SC(x,y) \rightarrow CO(x,y) \wedge \neg INCH(x,y) \wedge \neg INCH(y,x)$

Extensionality of *INCH* (I-PA1, I-PA2) and the Boolean operations *sum* and *diff* (I-PA9, I-PA10) are proper extensions of $T_{inch-basic}$, i.e. they are not entailed. Also I-PA8 does not follow, it is e.g. violated by entities of mixed dimensions such as a disk with a spike and needs to be added to restrict $T_{inch-basic}$ to the models of the INCH Calculus.

(I-PA1) $x = y \leftrightarrow \forall z(INCH(x,z) \leftrightarrow INCH(y,z))$

(I-PA2) $x = y \leftrightarrow \forall z(INCH(z,x) \leftrightarrow INCH(z,y))$

(I-PA8) $CH(x,y) \rightarrow CS(x,y)$

(I-PA9) $x =_{dim} y \rightarrow \exists z \forall w(INCH(z,w) \leftrightarrow INCH(x,w) \vee INCH(y,w))$ (z = sum(x,y))

(I-PA10) $x =_{dim} y \rightarrow \exists z \forall w(INCH(z,w) \leftrightarrow \exists v[INCH(v,w) \wedge CH(v,x) \wedge \neg OV(v,y)])$ (z = diff(x,y))

Now we can formalize the relationship between our theory and the original INCH-calculus, denoted by T'_{inch}.

Theorem 1. *Let* $T_{inch-full} = T^0_{full-linear} \cup$ *{I-PA1, I-PA2, I-PA8 – I-PA10} and* T'_{inch} *={I-PA1 – I-PA10, I-D1 – I-D9}. Then* T'_{inch} *faithfully interprets* $T_{inch-full}$*, i.e. every model* $\mathscr{M}$ *of* $T_{inch-full}$ *can be extended to a model* $\mathscr{M}'$ *of* T'_{inch} *so that all sentences consistent with* $\mathscr{M}$ *are consistent with* $\mathscr{M}'$.

The reverse does not hold, i.e. not every model of T'_{inch} is also a model of $T_{inch-full}$. This is due to the failure of the sentence (the inverse of the implication in I-M1)

(I-M1R) $CS(x,y) \wedge \neg ZEX(x) \rightarrow Cont(x,y)$

indicating that either: (a) we do not capture all intended interpretations of *INCH* or (b) that the axiomatization in (Gotts 1996) has unintended models. This necessitates a closer investigation beyond the scope of this paper. However, we do have the following partial result:

Theorem 2. $T_{inch-full} \cup$ *I-M1R entails the sentences*

(I-M3R) $C(x,y) \rightarrow CO(x,y)$

(I-M4R) $CH(x,y) \rightarrow P(x,y)$

(I-M5R) $EL(x,y) \wedge \neg ZEX(x) \rightarrow Cont(x,y) \wedge x < y$

(I-M7R) $CO(x,y) \wedge \neg INCH(x,y) \wedge \neg INCH(y,x) \rightarrow SC(x,y)$

Observe that I-M2 and I-M6 already contain the bidirectional implication and thus do not require additional inverses. Therefore, I-MIR establishes all equivalences between the relations of the INCH-calculus and the relations definable in our extension $T_{inch-full}$.

5.2 Equi-dimensional mereotopology (RCC)

In order to restrict the theory to models of equi-dimensional mereotopology, we cannot, somewhat counterintuitively, prohibit entities of lower dimensions. Otherwise, 'external connection' in the RCC or 'meet' in the Interval Calculus (Allen 1983), both special cases of *SC* (cf. SC-E1), have empty extensions according to SC-T4. This reduces the mereotopology to a pure mereology with overlap as only contact relation. Instead, we base the mapping to models of the RCC on the set of *non-zero regions*: the entities of maximum dimension (RCC-D1) guaranteed to exist by D-E2 in $T^0_{full-linear} \cup T_{dim-bounded}$. We do not attempt to directly construct models of the RCC, but instead show how we can obtain connected atomless Boolean contact algebras. The representation of RCC models by this class of contact algebras from (Düntsch & Winter 2005) then allows us to conclude that equivalent models of the RCC must exist.

(RCC-D1) $NZRegion(x) \leftrightarrow MaxDim(x)$ (non-zero regions)

(RCC-D2) $R(x) \leftrightarrow NZRegion(x) \vee ZEX(x)$ (regions)

(RCC-D3) $RP(x,y) \leftrightarrow (P(x,y) \wedge R(x)) \vee (ZEX(x) \wedge R(y))$ (parthood amongst regions)

We require binary meets (intersections, RCC-A1) and joins (sums, RCC-A2) and a universal (RCC-A3) in the set R resulting in bounded lattices $\langle R, RP \rangle$ with RP defining the partial order. These bounded lattices are Boolean because of unicomplementation (RCC-A4) and region-extensionality of C (RCC-A5). To obtain connected atomless Boolean contact algebras it now suffices to ensure connectedness (complements are in *SC*; RCC-A6). Then, all non-trivial models (RCC-A7) are automatically atomless by RCC-T1, cf. (Düntsch & Winter 2005).

(RCC-A1) $R(x) \wedge R(y) \rightarrow \exists m[R(m) \wedge RP(m,x) \wedge RP(m,y) \wedge \forall z(RP(z,x) \wedge RP(z,y) \leftrightarrow RP(z,m))]$ $(m = x \cdot y)$

(RCC-A2) $R(x) \wedge R(y) \rightarrow \exists j[R(j) \wedge RP(x,j) \wedge RP(y,j) \wedge \forall z(RP(x,z) \wedge RP(y,z) \leftrightarrow RP(j,z))]$ $(j = x + y)$

(RCC-A3) $R(U) \wedge \forall x(R(x) \rightarrow RP(x,U))$ (universal U)

(RCC-A4) $NZRegion(x) \wedge ZEX(z) \rightarrow [NZRegion(x') \wedge x + x' = U \wedge x \cdot x' = z \wedge \forall v[(NZRegion(v) \wedge x + v = U \wedge x \cdot v = z) \rightarrow x' = v]]$ (x' is the unique complement of x)

(RCC-A5) $R(x) \wedge R(y) \rightarrow [x = y \leftrightarrow \forall z[R(z) \rightarrow (C(z,x) \leftrightarrow C(z,y))]]$ (extensionality of C amongst regions)

(RCC-A6) $NZRegion(x) \wedge x \neq U \rightarrow SC(x,x')$ (connectedness)

(RCC-A7) $\exists x(NZRegion(x) \wedge x \neq U)$ (non-trivial)

(RCC-T1) $NZRegion(x) \rightarrow \exists y(x \neq y \wedge P(y,x))$ (inf. divisibility)

Theorem 3. *Suppose* $T^0_{dim-eq} = T^0_{full-linear} \cup T_{dim-bounded} \cup$ *{RCC-D1 – RCC-D3, RCC-A1 – RCC-A7}.*

For any model $\mathscr{M}$ *of* T^0_{dim-eq} *there exists a model* $\mathscr{N}$ *of RCC such that* $\mathscr{N}$ *is definably interpreted in* $\mathscr{M}$.

To prove Theorem 3, recall that a structure $\mathscr{N}$ with language $\mathscr{L}_1$ is definable in a structure $\mathscr{M}$ with language $\mathscr{L}_2$ iff there exists a definable set $X \subseteq M^n$ and we can interpret the symbols of $\mathscr{L}_1$ as definable subsets and functions on X so that the resulting structure is isomorphic to $\mathscr{N}$ (Grüninger *et al.* 2010). For the proof of Theorem 3 we choose $X = \{x | \langle \mathbf{x} \rangle \in \mathbf{NZRegion}\}$ as the definable set. Since we already ensured that X forms a connected atomless Boolean algebra, it remains to show that for all elements $x, y \in X$, contact C in $\mathscr{N}$ is defined as $C(x,y) \equiv \neg P(x,y')$ (as in RCC models) exactly when $C(x,y)$ holds in $\mathscr{M}$. This is guaranteed by the axioms C-D (definition of contact), P-D (definition of parthood), RCC-A4, RCC-A5, and RCC-A6. All other relations in the model $\mathscr{N}$ are in turn definable in terms of C (as in RCC). We thus obtain a connected, atomless Boolean contact algebra $\langle X, \mathbf{C} \rangle$ with the zero element removed, which is a model of the RCC.

6 Summary and outlook

We have presented an ontology that clearly separates properties of dimension from dimension-independent mereotopological primitives (containment and contact). We demonstrated how mereotopological relations, in particular parthood and the various contact relations, become definable through the rather weak interaction of dimension and containment. The dimension is only used relatively, that is, parthood as well as partial overlap require spatial entities to be of equal dimension, whereas incidence requires one element to be of lower dimension and weak contact is only applicable of the shared entity is of a lower dimension. Without using absolute dimensions (such as a two-dimensional entity), we can provide a jointly exhaustive, pairwise disjoint categorization of (binary) contact between spatial entities in terms of partial overlap, incidence, and weak contact.

We verified all discussed modules of the ontology by proving all theorems mentioned in the paper with an automated theorem prover and by generating non-trivial models for $T_{full} \cup \{$D-E1 to D-E4$\}$ (T_{full} restricted to linear and bounded dimensions) showing the consistency of the theory. Each non-trivial model ensures that the extensions of all spatial relations are non-empty. Then all weaker theories discussed in the paper also have non-trivial models. Moreover, we confirmed that commonsensical descriptions of the sketches in Fig. 1 are consistent with the ontology. The ontology is applicable in any domain dealing with spatial entities of two or more dimensions. We intend to explore the following application and refinements in the future.

- Interpret incidence geometry as an extension of our ontology by limiting the number of distinct dimensions.
- Refine superficial contact to capture intuitive notions such as 'touches', 'borders', 'crosses', and 'attachment'.
- Explore the reasoning capabilities and limitations of an extension capturing sketch maps (Reiter & Mackworth 1989). What inferences are possible and can the necessary number of distinct dimensions be determined without explicitly assigning dimensions to all map features, e.g. the river or the street in Fig. 1?
- Define artifacts in manufacturing such as cracks (cf. Hahmann & Grüninger 2009) in the ontology.

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